Wydział Mechaniczny Energetyki i Lotnictwa
Zakład Wytrzymałości Materiałów i Konstrukcji

Metoda elementów skończonych (MES1)

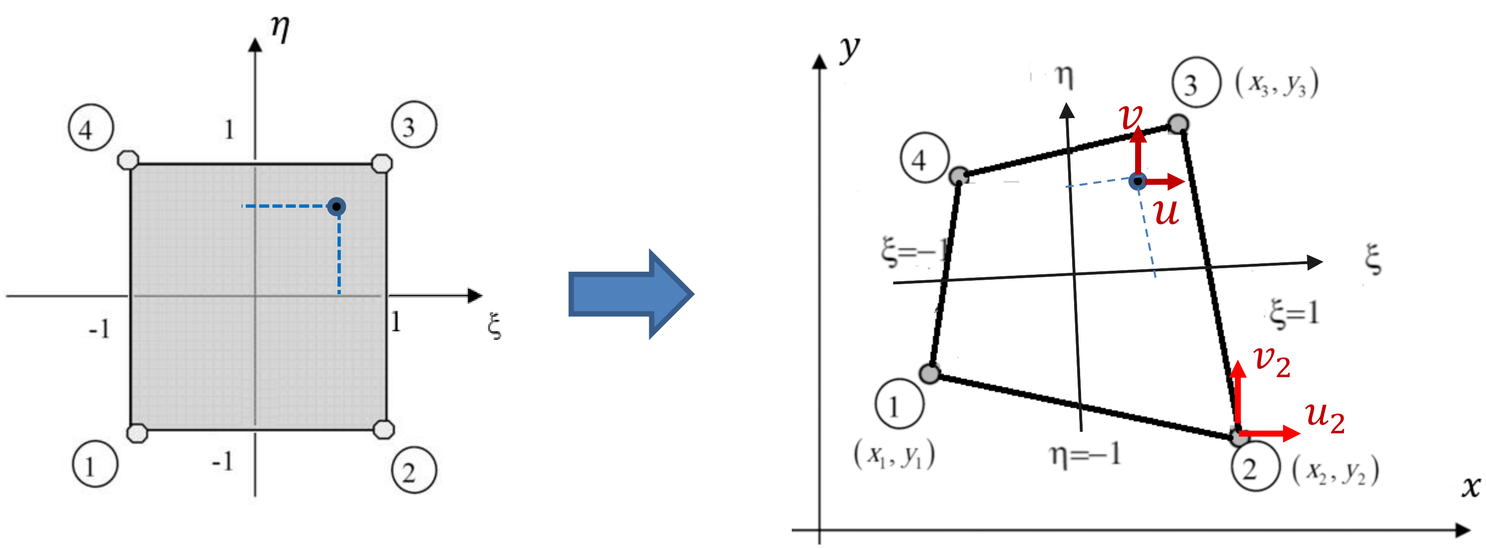
Wykład 6A. 4-węzłowy element czworokątny

03.2024

4-węzłowy 2D element czworokątny

układ współrzędnych naturalnych

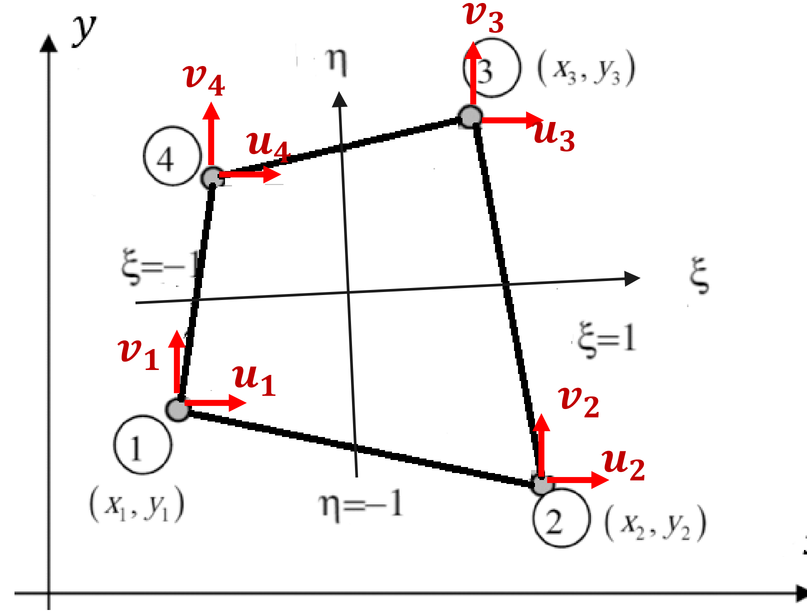
układ współrzędnych kartezjańskich



Odwzorowanie geometryczne: $\boxed{(\xi, \eta) \rightarrow (x, y)}$

$$(-1,-1) \rightarrow (x_1, y_1) \qquad (1,-1) \rightarrow (x_2, y_2) \qquad (1,1) \rightarrow (x_3, y_3) \qquad (-1,1) \rightarrow (x_4, y_4)$$

Odwzorowanie izoparametryczne

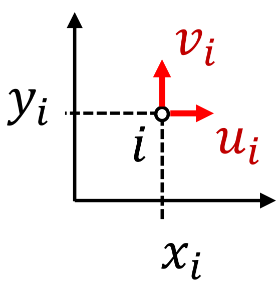


$n = 4 \;\; ; \;\; n_p = 2 \;\; \rightarrow \;\; n_e = n \cdot n_p = 8$

wektory współrzędnych węzłowych

$$\underset{4\times 1}{\{x_i\}_e} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}; \quad \underset{4\times 1}{\{y_i\}_e} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix}$$

wektor lokalny parametrów węzłowych:



$$\underset{8\times 1}{\{q\}_e} = \begin{Bmatrix} q_1 \\ q_2 \\ \cdot \\ \cdot \\ \cdot \\ q_8 \end{Bmatrix}_e = \begin{Bmatrix} u_1 \\ v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_4 \end{Bmatrix}_e$$

Odwzorowanie izoparametryczne
–te same funkcje użyte są do opisu geometrii i pola przemieszczeń

Odwzorowanie izoparametryczne

skonstruujemy funkcje kształtu:

$$x = a + b\cdot\xi + c\cdot\eta + d\cdot\xi\eta = \lfloor 1, \xi, \eta, \xi\eta \rfloor \cdot \begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix}$$

a, b, c, d – stałe

$$(\xi,\eta) \qquad\qquad (x,y)$$

① $(-1,-1) \rightarrow (x_1, y_1)$ $\qquad x_1 = 1\cdot a + (-1)\cdot b + (-1)\cdot c + (-1)\cdot(-1)\cdot d$

② $(1,-1) \rightarrow (x_2, y_2)$ $\qquad x_2 = 1\cdot a + 1\cdot b + (-1)\cdot c + 1\cdot(-1)\cdot d$

③ $(1,1) \rightarrow (x_3, y_3)$ $\qquad x_3 = 1\cdot a + 1\cdot b + 1\cdot c + 1\cdot 1\cdot d$

④ $(-1,1) \rightarrow (x_4, y_4)$ $\qquad x_4 = 1\cdot a + (-1)\cdot b + 1\cdot c + (-1)\cdot 1\cdot d$

$$\{x_i\}_e = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}_e = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \cdot \begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix} = [A] \cdot \begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix}$$

$(\det[A] = -16)$

$$\begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix} = [A]^{-1} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}_e = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix} \cdot \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}_e$$

$$x = \lfloor 1, \xi, \eta, \xi\eta \rfloor \begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix} = \lfloor 1, \xi, \eta, \xi\eta \rfloor \cdot \underset{4\times 4}{[A]^{-1}} \cdot \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}_e =$$

$$
= \lfloor 1, \xi, \eta, \xi\cdot\eta \rfloor \cdot \begin{Bmatrix} \frac{1}{4}x_1 + \frac{1}{4}x_2 + \frac{1}{4}x_3 + \frac{1}{4}x_4 \\ -\frac{1}{4}x_1 + \frac{1}{4}x_2 + \frac{1}{4}x_3 - \frac{1}{4}x_4 \\ -\frac{1}{4}x_1 - \frac{1}{4}x_2 + \frac{1}{4}x_3 + \frac{1}{4}x_4 \\ \frac{1}{4}x_1 - \frac{1}{4}x_2 + \frac{1}{4}x_3 - \frac{1}{4}x_4 \end{Bmatrix} =
$$

$$
= 1\cdot\left(\frac{1}{4}x_1 + \frac{1}{4}x_2 + \frac{1}{4}x_3 + \frac{1}{4}x_4\right) + \xi\cdot\left(-\frac{1}{4}x_1 + \frac{1}{4}x_2 + \frac{1}{4}x_3 - \frac{1}{4}x_4\right) +
$$

$$
+ \eta\left(-\frac{1}{4}x_1 - \frac{1}{4}x_2 + \frac{1}{4}x_3 + \frac{1}{4}x_4\right) + \xi\eta\left(\frac{1}{4}x_1 - \frac{1}{4}x_2 + \frac{1}{4}x_3 - \frac{1}{4}x_4\right) =
$$

$$
= \left(\frac{1}{4} - \frac{1}{4}\xi - \frac{1}{4}\eta + \frac{1}{4}\xi\eta\right)x_1 + \left(\frac{1}{4} + \frac{1}{4}\xi - \frac{1}{4}\eta - \frac{1}{4}\xi\eta\right)x_2 +
$$

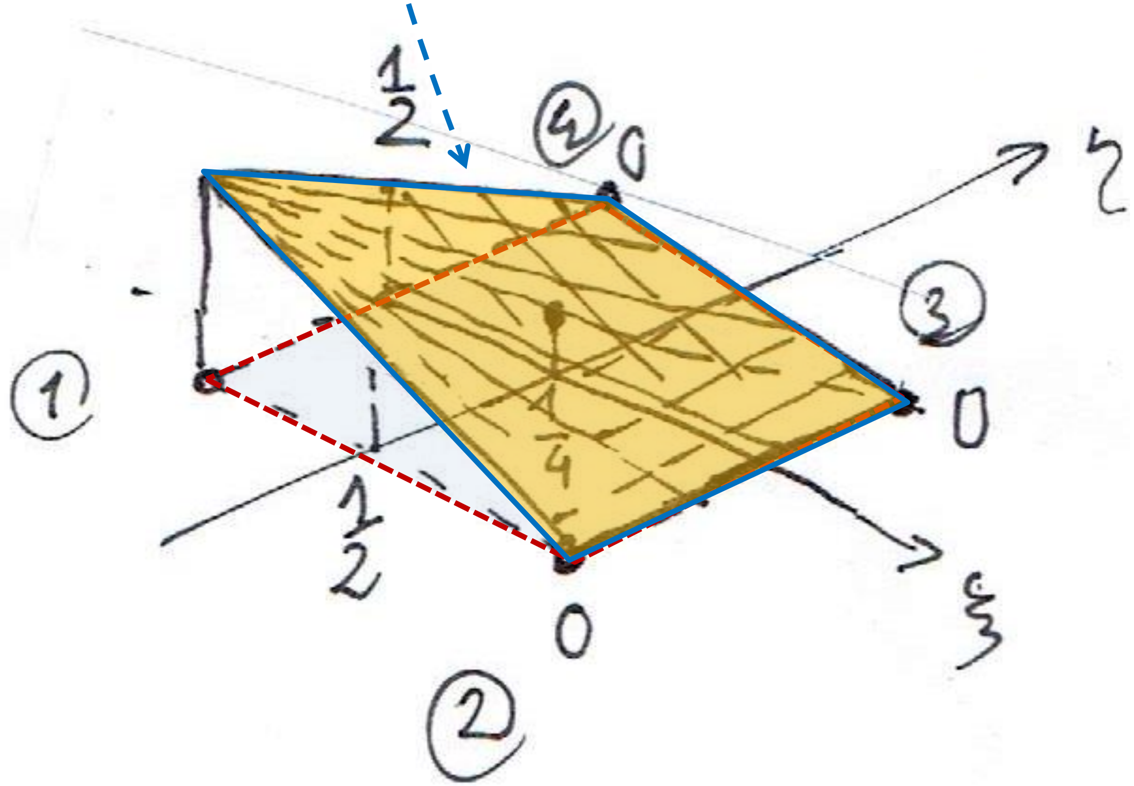
$$
+ \left(\frac{1}{4} + \frac{1}{4}\xi + \frac{1}{4}\eta + \frac{1}{4}\xi\eta\right)x_3 + \left(\frac{1}{4} - \frac{1}{4}\xi + \frac{1}{4}\eta - \frac{1}{4}\xi\eta\right)\cdot x_4 =
$$

$$
= \boxed{\frac{(1-\xi)(1-\eta)}{4}}\cdot x_1 + \boxed{\frac{(1+\xi)(1-\eta)}{4}}\cdot x_2 + \boxed{\frac{(1+\xi)(1+\eta)}{4}}\cdot x_3 + \boxed{\frac{(1-\xi)(1+\eta)}{4}}\cdot x_4 =
$$

$$
= N_1(\xi,\eta)\cdot x_1 + N_2(\xi,\eta)\cdot x_2 + N_3(\xi,\eta)x_3 + N_4(\xi,\eta)\cdot x_4
$$

Funkcje kształtu elementu 4 węzłowego

$$N_1(\xi,\eta) = \tfrac{1}{4}(1-\xi)(1-\eta)$$
$$N_2(\xi,\eta) = \tfrac{1}{4}(1+\xi)(1-\eta)$$
$$N_3(\xi,\eta) = \tfrac{1}{4}(1+\xi)(1+\eta)$$
$$N_4(\xi,\eta) = \tfrac{1}{4}(1-\xi)(1+\eta)$$



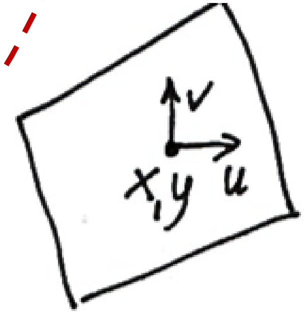
Odwzorowanie izoparametryczne

Odwzorowanie izoparametryczne –te same funkcje użyte są do opisu geometrii i pola przemieszczeń

$$x = \underset{1\times 4}{\lfloor N(\xi,\eta) \rfloor} \cdot \underset{4\times 1}{\{x_i\}_e}$$

$$y = \underset{1\times 4}{\lfloor N(\xi,\eta) \rfloor} \cdot \underset{4\times 1}{\{y_i\}_e}$$

$$\underset{2\times 1}{\{u\}} = \begin{Bmatrix} u \\ v \end{Bmatrix} = \underset{2\times 8}{[N(\xi,\eta)]} \cdot \underset{8\times 1}{\{q\}_e}$$



położenie i przemieszczenie dowolnego punktu

gdzie:

$$\underset{1\times 4}{\lfloor N(\xi,\eta) \rfloor} = \lfloor N_1, N_2, N_3, N_4 \rfloor$$

$$\underset{2\times 8}{[N(\xi,\eta)]} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}$$

Operatory różniczkowe w układzie naturalnym

$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial x} \cdot \frac{\partial x}{\partial \xi} + \frac{\partial}{\partial y} \frac{\partial y}{\partial \xi}$$

$$\frac{\partial}{\partial \eta} = \frac{\partial}{\partial x} \cdot \frac{\partial x}{\partial \eta} + \frac{\partial}{\partial y} \frac{\partial y}{\partial \eta}$$

$$\Rightarrow \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}}_{[J]} \cdot \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix}$$

$[J]$ – Macierz Jacobiego

Operatory różniczkowe w układzie kartezjańskim

$$\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = [J]^{-1} \cdot \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = \frac{1}{\det[J]} \left(\underset{2\times 2}{[J]^{C}} \right)^{T} \cdot \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix}$$

$$\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = \begin{bmatrix} \frac{1}{\det[J]} \cdot \frac{\partial y}{\partial \eta} & -\frac{1}{\det[J]} \cdot \frac{\partial y}{\partial \xi} \\ -\frac{1}{\det[J]} \cdot \frac{\partial x}{\partial \eta} & \frac{1}{\det[J]} \frac{\partial x}{\partial \xi} \end{bmatrix} \cdot \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix}$$

$$\det[J] = \frac{\partial x}{\partial \xi}\cdot\frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi}\cdot\frac{\partial x}{\partial \eta} \Rightarrow$$

$$\boxed{\frac{\partial x}{\partial \xi}} = \frac{\partial\left(\underset{1\times4}{\lfloor N(\xi,\eta)\rfloor}\cdot\underset{4\times1}{\{x_i\}_e}\right)}{\partial \xi} = \frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \xi}\cdot\underset{4\times1}{\{x_i\}_e} + \underbrace{\frac{\partial\{x_i\}_e}{\partial \xi}}_{=0}\cdot\underset{1\times4}{\lfloor N(\xi,\eta)\rfloor} =$$

(wartości dyskretne)

$$= \left\lfloor \frac{\partial N_1}{\partial \xi}, \frac{\partial N_2}{\partial \xi}, \frac{\partial N_3}{\partial \xi}, \frac{\partial N_4}{\partial \xi}\right\rfloor \cdot \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}_e =$$

$$= \boxed{\left(-\frac{1}{4}(1-\eta)\right)\cdot x_1 + \frac{1}{4}(1-\eta)\cdot x_2 + \frac{1}{4}(1+\eta)x_3 - \frac{1}{4}(1+\eta)x_4}$$

$$\boxed{\frac{\partial y}{\partial \eta}} = \frac{\partial\left(\underset{1\times4}{\lfloor N(\xi,\eta)\rfloor}\cdot\underset{4\times1}{\{y_i\}_e}\right)}{\partial \eta} = \frac{\partial\underset{1\times4}{\lfloor N(\xi,\eta)\rfloor}}{\partial \eta}\cdot\underset{4\times1}{\{y_i\}_e} =$$

$$= \boxed{\left(-\frac{1}{4}(1-\xi)\right)\cdot y_1 - \frac{1}{4}(1+\xi)\cdot y_2 + \frac{1}{4}(1+\xi)\cdot y_3 + \frac{1}{4}(1-\xi)\cdot y_4}$$

$$\boxed{\frac{\partial y}{\partial \xi}} = \frac{\partial\left(\lfloor N(\xi,\eta)\rfloor_{1\times 4}\cdot\{y_i\}_{4\times 1}^{e}\right)}{\partial \xi} = \frac{\partial \lfloor N(\xi,\eta)\rfloor_{1\times 4}}{\partial \xi}\cdot\{y_i\}_{4\times 1}^{e} =$$

$$= \boxed{\left(-\tfrac{1}{4}(1-\eta)\right)\cdot y_1 + \tfrac{1}{4}(1-\eta)\cdot y_2 + \tfrac{1}{4}(1+\eta)y_3 - \tfrac{1}{4}(1+\eta)y_4}$$

$$\boxed{\frac{\partial x}{\partial \eta}} = \frac{\partial\left(\lfloor N(\xi,\eta)\rfloor_{1\times 4}\cdot\{x_i\}_{4\times 1}^{e}\right)}{\partial \eta} = \frac{\partial \lfloor N(\xi,\eta)\rfloor_{1\times 4}}{\partial \eta}\cdot\{x_i\}_{4\times 1}^{e} =$$

$$= \boxed{\left(-\tfrac{1}{4}(1-\xi)\right)x_1 - \tfrac{1}{4}(1+\xi)\cdot x_2 + \tfrac{1}{4}(1+\xi)x_3 + \tfrac{1}{4}(1-\xi)\cdot x_4}$$

$$\frac{\partial}{\partial x} = \frac{1}{\det[J]}\left(\frac{\partial y}{\partial \eta}\cdot\frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi}\cdot\frac{\partial}{\partial \eta}\right)$$

$$\frac{\partial}{\partial y} = \frac{1}{\det[J]}\left(\frac{\partial x}{\partial \xi}\cdot\frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta}\cdot\frac{\partial}{\partial \xi}\right)$$

Macierz gradientów dla warunku PSN lub PSO

$$\underset{3\times 2}{\left[R(x,y)\right]} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} = \underbrace{\frac{1}{\det[J]} \begin{bmatrix} \frac{\partial y}{\partial \eta}\cdot\frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi}\cdot\frac{\partial}{\partial \eta} & 0 \\ 0 & \frac{\partial x}{\partial \xi}\cdot\frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta}\cdot\frac{\partial}{\partial \xi} \\ \frac{\partial x}{\partial \xi}\cdot\frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta}\cdot\frac{\partial}{\partial \xi} & \frac{\partial y}{\partial \eta}\cdot\frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi}\cdot\frac{\partial}{\partial \eta} \end{bmatrix}}_{\underset{3\times 2}{\left[R(\xi,\eta)\right]}}$$

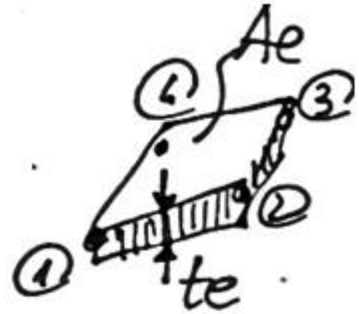
Wektor składowych odkształcenia (PSN, PSO)

$$\underset{3\times 1}{\{\varepsilon\}} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \underset{3\times 2}{\left[R(\xi,\eta)\right]} \cdot \underset{2\times 1}{\{u\}} = \underset{3\times 2}{\left[R(\xi,\eta)\right]} \cdot \underset{2\times 8}{\left[N(\xi,\eta)\right]} \cdot \underset{8\times 1}{\{q\}_e} = \underset{3\times 8}{\left[B(\xi,\eta)\right]} \cdot \underset{8\times 1}{\{q\}_e}$$

Wektor składowych naprężenia (PSN, PSO)

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \underset{3\times 3}{[D]} \cdot \underset{3\times 1}{\{\varepsilon\}} = \underset{3\times 3}{[D]} \cdot \underset{3\times 8}{\left[B(\xi,\eta)\right]} \cdot \underset{8\times 1}{\{q\}_e}$$

Energia odkształcenia sprężystego w elemencie skończonym



$$U_e = \frac{1}{2}\int_{\Omega_e} \underset{1\times 3}{\lfloor \varepsilon \rfloor} \underset{3\times 1}{\{\sigma\}} \, d\Omega_e = \frac{1}{2} t_e \int_{A_e} \underset{1\times 3}{\lfloor \varepsilon \rfloor} \cdot \underset{3\times 1}{\{\sigma\}} \, dA_e =$$

$$= \frac{1}{2} t_e \int_{-1}^{1}\int_{-1}^{1} \lfloor \varepsilon \rfloor \cdot \{\sigma\} \det[J] \, d\xi \, d\eta = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \cdot \underset{8\times 8}{[k]_e} \underset{8\times 1}{\{q\}_e}$$

gdzie:

$$[k]_e = t_e \underbrace{\int_{-1}^{1}\int_{-1}^{1} \left(\underset{8\times 3}{[B(\xi,\eta)]^T} \underset{3\times 3}{[D]} \cdot \underset{3\times 8}{[B(\xi,\eta)]} \det[J(\xi,\eta)] \right) d\xi \, d\eta}_{\text{(całkowanie numeryczne)}}$$

$$\underset{3\times 2}{[R(x,y)]} = \frac{1}{\det[J]} \begin{bmatrix} \frac{\partial y}{\partial \eta}\cdot\frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi}\cdot\frac{\partial}{\partial \eta} & 0 \\ 0 & \frac{\partial x}{\partial \xi}\cdot\frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta}\cdot\frac{\partial}{\partial \xi} \\ \frac{\partial x}{\partial \xi}\cdot\frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta}\cdot\frac{\partial}{\partial \xi} & \frac{\partial y}{\partial \eta}\cdot\frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi}\cdot\frac{\partial}{\partial \eta} \end{bmatrix}$$

$$\underset{3\times 8}{[B(\xi,\eta)]} = \underset{3\times 2}{[R(\xi,\eta)]}\cdot\underset{2\times 8}{[N(\xi,\eta)]} = \begin{bmatrix} r_{11} & r_{12}=0 \\ r_{21}=0 & r_{22} \\ \underset{r_{22}}{\overset{\parallel}{r_{31}}} & \underset{r_{11}}{\overset{\parallel}{r_{32}}} \end{bmatrix} \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}$$

$$r_{11}\cdot N_1 = \frac{\frac{\partial y}{\partial \eta}\cdot\frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi}\cdot\frac{\partial}{\partial \eta}}{\det[J]}\cdot N_1(\xi,\eta) = \frac{1}{\det[J]}\cdot\left(\frac{\partial y}{\partial \eta}\cdot\frac{\partial N_1}{\partial \xi} - \frac{\partial y}{\partial \xi}\cdot\frac{\partial N_1}{\partial \eta}\right) =$$

$$= \frac{-\frac{1}{4}(1-\eta)\frac{\partial y}{\partial \eta} + \frac{1}{4}(1-\xi)\frac{\partial y}{\partial \xi}}{\det[J]} = b_{11} = b_{32}$$

$$r_{11}\cdot N_2 = \frac{\frac{1}{4}(1-\eta)\frac{\partial y}{\partial \eta} + \frac{1}{4}(1+\xi)\frac{\partial y}{\partial \xi}}{\det[J]} = b_{13} = b_{34}$$

$$r_{11}\cdot N_3 = \frac{\frac{1}{4}(1+\eta)\frac{\partial y}{\partial \eta} - \frac{1}{4}(1+\xi)\frac{\partial y}{\partial \xi}}{\det[J]} = b_{15} = b_{36}$$

$$r_{11}\cdot N_4 = \frac{-\frac{1}{4}(1+\eta)\frac{\partial y}{\partial \eta} - \frac{1}{4}(1-\xi)\frac{\partial y}{\partial \xi}}{\det[J]} = b_{17} = b_{38}$$

$$\underset{3\times 2}{[R(x,y)]} = \frac{1}{\det[J]}\begin{bmatrix} \frac{\partial y}{\partial \eta}\cdot\frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi}\cdot\frac{\partial}{\partial \eta} & 0 \\ 0 & \frac{\partial x}{\partial \xi}\cdot\frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta}\cdot\frac{\partial}{\partial \xi} \\ \frac{\partial x}{\partial \xi}\cdot\frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta}\cdot\frac{\partial}{\partial \xi} & \frac{\partial y}{\partial y}\cdot\frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi}\cdot\frac{\partial}{\partial \eta} \end{bmatrix}$$

$$r_{22}\cdot N_1 = \frac{\frac{\partial x}{\partial \xi}\cdot\frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial}{\partial \xi}}{\det[J]}\cdot N_1(\xi,\eta) \quad = \frac{1}{\det[J]}\left(\frac{\partial x}{\partial \xi}\cdot\frac{\partial N_1}{\partial \eta} - \frac{\partial x}{\partial \eta}\cdot\frac{\partial N_1}{\partial \xi}\right) =$$

$$= \frac{-\frac{1}{4}(1-\xi)\frac{\partial x}{\partial \xi} + \frac{1}{4}(1-\eta)\frac{\partial x}{\partial \eta}}{\det[J]} = b_{22} = b_{31}$$

$$r_{22}\cdot N_2 = \frac{-\frac{1}{4}(1+\xi)\frac{\partial x}{\partial \xi} - \frac{1}{4}(1-\eta)\frac{\partial x}{\partial \eta}}{\det[J]} = b_{24} = b_{33}$$

$$r_{22}\cdot N_3 = \frac{\frac{1}{4}(1+\xi)\frac{\partial x}{\partial \xi} - \frac{1}{4}(1+\eta)\frac{\partial x}{\partial \eta}}{\det[J]} = b_{26} = b_{35}$$

$$r_{22}\cdot N_4 = \frac{\frac{1}{4}(1-\xi)\frac{\partial x}{\partial \xi} + \frac{1}{4}(1+\eta)\frac{\partial x}{\partial \eta}}{\det[J]} = b_{28} = b_{37}$$

$$\underset{3\times 8}{[B]} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} & b_{17} & b_{18} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} & b_{27} & b_{28} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & b_{36} & b_{37} & b_{38} \end{bmatrix}$$

Rozbicie energii odkształcenia sprężystego na tę związaną z naprężeniami normalnymi i tę od ścinania

$$U_e = \frac{1}{2}\int_{\Omega_e} \lfloor \varepsilon_x, \varepsilon_y, \gamma_{xy} \rfloor \cdot [D] \cdot \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} d\Omega_e =$$

$$= \underbrace{\frac{1}{2}\int_{\Omega_e} \lfloor \varepsilon_x, \varepsilon_y, 0 \rfloor \underset{3\times 3}{[D]} \cdot \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ 0 \end{Bmatrix} d\Omega_e}_{U_e^{\sigma} \text{ (normal stress)}} + \underbrace{\frac{1}{2}\int_{\Omega_e} \lfloor 0, 0, \gamma_{xy} \rfloor \cdot \underset{3\times 3}{[D]} \cdot \begin{Bmatrix} 0 \\ 0 \\ \gamma_{xy} \end{Bmatrix} d\Omega_e}_{U_e^{\tau} \text{ (shear stress)}}$$

$$\underset{3\times 1}{\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ 0 \end{Bmatrix}} = \underset{3\times 2}{\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ 0 & 0 \end{bmatrix}} \underset{2\times 1}{\{u\}} = \underset{3\times 2}{\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ 0 & 0 \end{bmatrix}} \cdot \underset{2\times 8}{[N(\xi,\eta)]} \cdot \underset{8\times 1}{\{q\}_e} = \underset{3\times 8}{[B_\varepsilon]} \cdot \underset{8\times 1}{\{q\}_e}$$

$$\underset{3\times 1}{\begin{Bmatrix} 0 \\ 0 \\ \gamma_{xy} \end{Bmatrix}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \cdot \underset{2\times 1}{\{u\}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \underset{2\times 8}{[N(\xi,\eta)]} \underset{8\times 1}{\{q\}_e} = \underset{3\times 8}{[B_\gamma]} \cdot \underset{8\times 1}{\{q\}_e}$$

$$[B_\varepsilon] = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{18} \\ b_{21} & b_{22} & \dots & b_{28} \\ 0 & 0 & \dots & 0 \end{bmatrix} \qquad [B_\gamma] = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ b_{31} & b_{32} & \dots & b_{38} \end{bmatrix}$$

Rozbicie energii odkształcenia sprężystego na tę związaną z naprężeniami normalnymi i tę od ścinania

$$U_e^{\sigma} = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \underset{8\times 8}{[k_\varepsilon]_e} \underset{8\times 1}{\{q\}_e}$$

$$U_e^{\tau} = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \underset{8\times 8}{[k_\gamma]_e} \underset{8\times 1}{\{q\}_e}$$

gdzie:

$$[k_\varepsilon]_e = t_e \int_{-1}^{1}\int_{-1}^{1} \left([B_\varepsilon]^T [D] [B_\varepsilon] \cdot \det[J(\xi,\eta)] \right) d\xi\, d\eta$$

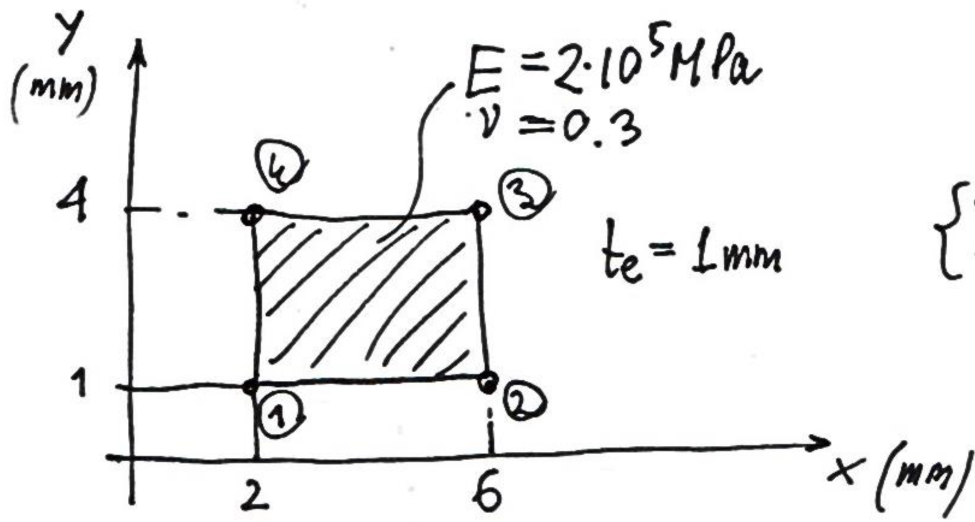
$$[k_\gamma]_e = t_e \int_{-1}^{1}\int_{-1}^{1} \left([B_\gamma]^T [D] [B_\gamma] \det[J(\xi,\eta)] \right) d\xi\, d\eta$$

$$[k]_e = [k_\varepsilon]_e + [k_\gamma]_e$$

Macierz sztywności związana z odkształceniami kątowymi

Macierz sztywności związana z odkształceniami liniowymi

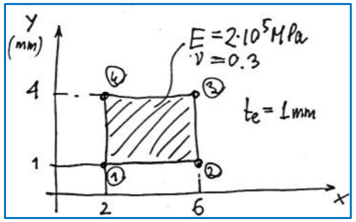
Przykład 4-węzłowy element prostokątny



$t_e = 1$ mm

$$\{x_i\}_e = \begin{Bmatrix} 2 \\ 6 \\ 6 \\ 2 \end{Bmatrix} \; ; \quad \{y_i\}_e = \begin{Bmatrix} 1 \\ 1 \\ 4 \\ 4 \end{Bmatrix}$$

$$x_4 = x_1 \, , \quad x_3 = x_2 \, , \quad y_2 = y_1 \; ; \quad y_4 = y_3$$



$$\frac{\partial x}{\partial \xi} = \left(-\frac{1}{4}(1-\eta)\right)\cdot x_1 + \frac{1}{4}(1-\eta)x_2 + \frac{1}{4}(1+\eta)x_3 - \frac{1}{4}(1+\eta)x_4 =$$

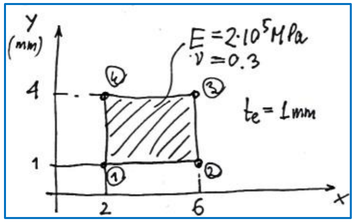
$$= \left(-\frac{1}{4}(1-\eta) - \frac{1}{4}(1+\eta)\right)\cdot x_1 + \left(\frac{1}{4}(1-\eta) + \frac{1}{4}(1+\eta)\right)x_2 =$$

$$= -\frac{1}{2}x_1 + \frac{1}{2}x_2 = \frac{1}{2}(x_2 - x_1) = \frac{1}{2}(6-2) = 2\,\text{mm}$$

$$\frac{\partial y}{\partial \eta} = \left(-\frac{1}{4}(1-\xi)\right)\cdot y_1 - \frac{1}{4}(1+\xi)\cdot y_2 + \frac{1}{4}(1+\xi)\cdot y_3 + \frac{1}{4}(1-\xi)\cdot y_4 =$$

$$= \left(-\frac{1}{4}(1-\xi) - \frac{1}{4}(1+\xi)\right)y_1 + \left(\frac{1}{4}(1+\xi) + \frac{1}{4}(1-\xi)\right)y_3 =$$

$$= -\frac{1}{2}y_1 + \frac{1}{2}y_3 = \frac{1}{2}(y_3 - y_1) = \frac{1}{2}(4-1) = 1.5\,\text{mm}$$



$$\frac{\partial y}{\partial \xi} = \left(-\frac{1}{4}(1-\eta)\right)\cdot y_1 + \frac{1}{4}(1-\eta)\cdot \overset{(y_1)}{\overset{\parallel}{y_2}} + \frac{1}{4}(1+\eta)\cdot y_3 - \frac{1}{4}(1+\eta)\cdot \overset{(y_3)}{\overset{\parallel}{y_4}} =$$

$$= 0\cdot y_1 + 0\cdot y_3 = 0$$

$$\frac{\partial x}{\partial \eta} = \left(-\frac{1}{4}(1-\xi)\right)\cdot x_1 - \frac{1}{4}(1+\xi)\cdot x_2 + \frac{1}{4}(1+\xi)\cdot \overset{(x_2)}{\overset{\parallel}{x_3}} + \frac{1}{4}(1-\xi)\cdot \overset{(x_1)}{\overset{\parallel}{x_4}} =$$

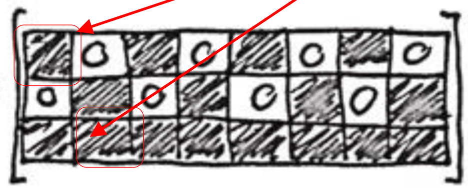
$$= 0\cdot x_1 + 0\cdot x_2 = 0$$

$$\boxed{\det[J] = \frac{\partial x}{\partial \xi}\cdot\frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi}\cdot\frac{\partial x}{\partial \eta} = 2\,mm\cdot 1.5\,mm - 0\cdot 0 = 3\,mm^2}$$

Macierz odkształcenie przemieszczenie:

$$\frac{-\frac{1}{4}(1-\eta)\frac{\partial y}{\partial \eta}+\frac{1}{4}(1-\xi)\frac{\partial y}{\partial \xi}}{\det[J]} = b_{11} = b_{32}$$

$$\underset{3\times 8}{[B]} =$$



$$b_{12} = b_{14} = b_{16} = b_{18} = b_{21} = b_{23} = b_{25} = b_{27} = 0$$

$$b_{11} = b_{32} = \frac{-\frac{1}{4}(1-\eta)\cdot 1.5mm + \frac{1}{4}(1-\xi)\cdot 0mm}{3\,mm^2} = -\frac{1}{8}(1-\eta)\,\frac{1}{mm}$$

$$b_{13} = b_{34} = \frac{\frac{1}{4}(1-\eta)\cdot 1.5mm + \frac{1}{4}(1+\xi)\cdot 0mm}{3mm^2} = \frac{1}{8}(1-\eta)\,\frac{1}{mm}$$

$$b_{15} = b_{36} = \frac{\frac{1}{4}(1+\eta)\cdot 1.5mm - \frac{1}{4}(1+\xi)\cdot 0mm}{3mm^2} = \frac{1}{8}(1+\eta)\,\frac{1}{mm}$$

$$b_{17} = b_{38} = \frac{-\frac{1}{4}(1+\eta)\cdot 1.5mm - \frac{1}{4}(1-\xi)\cdot 0mm}{3mm^2} = -\frac{1}{8}(1+\eta)\,\frac{1}{mm}$$

Macierz odkształcenie przemieszczenie:

$$b_{22} = b_{31} = \frac{-\frac{1}{4}(1-\xi)\cdot 2mm + \frac{1}{4}(1-\eta)\cdot 0mm}{3mm^2} = -\frac{1}{6}(1-\xi)\ \frac{1}{mm}$$

$$b_{24} = b_{33} = \frac{-\frac{1}{4}(1+\xi)\cdot 2mm - \frac{1}{4}(1-\eta)\cdot 0mm}{3mm^2} = -\frac{1}{6}(1+\xi)\ \frac{1}{mm}$$

$$b_{26} = b_{35} = \frac{\frac{1}{4}(1+\xi)\cdot 2mm - \frac{1}{4}(1+\eta)\cdot 0mm}{3mm^2} = \frac{1}{6}(1+\xi)\ \frac{1}{mm}$$

$$b_{28} = b_{37} = \frac{\frac{1}{4}(1-\xi)\cdot 2mm + \frac{1}{4}(1+\eta)\cdot 0mm}{3mm^2} = \frac{1}{6}(1-\xi)\ \frac{1}{mm}$$

$$\underset{3\times 8}{\left[B(\xi,\eta)\right]} = \begin{bmatrix} -(1-\eta)/8 & 0 & (1-\eta)/8 & 0 & (1+\eta)/8 & 0 & -(1+\eta)/8 & 0 \\ 0 & -(1-\xi)/6 & 0 & -(1+\xi)/6 & 0 & (1+\xi)/6 & 0 & (1-\xi)/6 \\ -(1-\xi)/6 & -(1-\eta)/8 & -(1+\xi)/6 & (1-\eta)/8 & (1+\xi)/6 & (1+\eta)/8 & (1-\xi)/6 & -(1+\eta)/8 \end{bmatrix}$$

$$\frac{1}{mm}$$

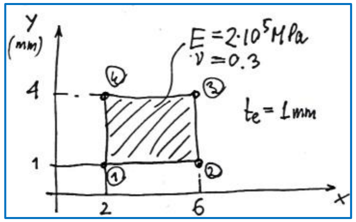
Macierz zawiera liniowe człony ze względu na współrzędne naturalne

Macierz odkształcenie przemieszczenie:

$$\underset{3\times 8}{\left[B(\xi,\eta)\right]} = \underset{3\times 8}{\left[B_{\varepsilon}(\xi,\eta)\right]} + \underset{3\times 8}{\left[B_{\gamma}(\xi,\eta)\right]}$$

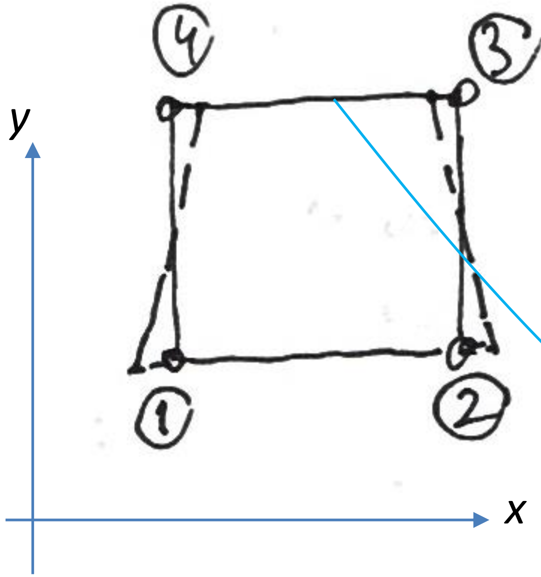
$$\underset{3\times 8}{\left[B_{\varepsilon}(\xi,\eta)\right]} = \begin{bmatrix} -(1-\eta)/8 & 0 & (1-\eta)/8 & 0 & (1+\eta)/8 & 0 & -(1+\eta)/8 & 0 \\ 0 & -(1-\xi)/6 & 0 & -(1+\xi)/6 & 0 & (1+\xi)/6 & 0 & (1-\xi)/6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \frac{1}{mm}$$

$$\underset{3\times 8}{\left[B_{\gamma}(\xi,\eta)\right]} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -(1-\xi)/6 & -(1-\eta)/8 & -(1+\xi)/6 & (1-\eta)/8 & (1+\xi)/6 & (1+\eta)/8 & (1-\xi)/6 & -(1+\eta)/8 \end{bmatrix} \frac{1}{mm}$$



Przypadek 1. *Rozważmy deformację „Zginanie” w elemencie 4w*

Wektor parametrów węzłowych:



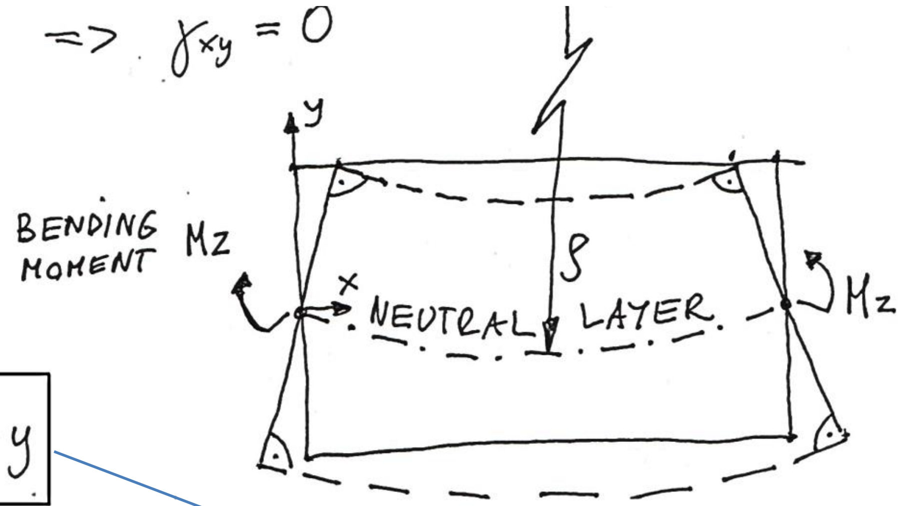
$$\underset{\substack{8\times 1\\(mm)}}{\{q\}_e} = \begin{array}{cc} \left\{\begin{matrix} -0.001 \\ 0 \\ 0.001 \\ 0 \\ -0.001 \\ 0 \\ 0.001 \\ 0 \end{matrix}\right\}_e & \begin{matrix} (u_1) \\ \\ (u_2) \\ \\ (u_3) \\ \\ (u_4) \\ \\ \end{matrix} \end{array}$$

$$\{x_i\}_e = \begin{Bmatrix} 2 \\ 6 \\ 6 \\ 2 \end{Bmatrix} \quad i \quad \{y_i\}_e = \begin{Bmatrix} 1 \\ 1 \\ 4 \\ 4 \end{Bmatrix}$$

Dla górnej warstwy mamy ε_x **= - 0.5·10^{-3}**

Teoria belkowa: Czyste zginanie belki (bez ścinania)

$\tau_{xy} = 0 \Rightarrow \gamma_{xy} = 0$

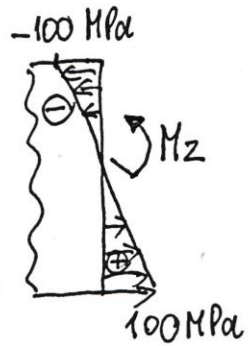


$$\boxed{\varepsilon_x = -\frac{1}{\rho}\cdot y}$$

Dla **y=1.5** mm mamy $\boldsymbol{\varepsilon_x = -0.5\cdot 10^{-3}}$

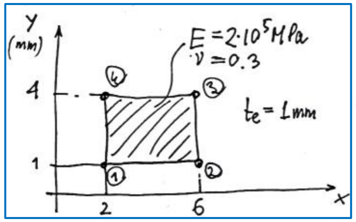
$$\rho = \frac{1.5}{0.5}\cdot 10^3\,mm = 3000\,mm$$

$$\boxed{\sigma_x = E\,\varepsilon_x} = -\frac{E}{\rho}\cdot y = -\frac{2\cdot 10^5\,MPa}{3000\,mm}\cdot y = -\frac{200}{3}\,\frac{MPa}{mm}\cdot y$$



$$\sigma_x(1.5mm) = -\frac{200}{3}\cdot 1.5\,MPa = -100\,MPa$$

$$\boxed{\sigma_x = -\frac{M_z\cdot y}{J_z}} \Rightarrow -\frac{M_z}{J_z} = -\frac{E}{\rho} \Rightarrow \boxed{\frac{1}{\rho} = \frac{M_z}{EJ_z}}$$

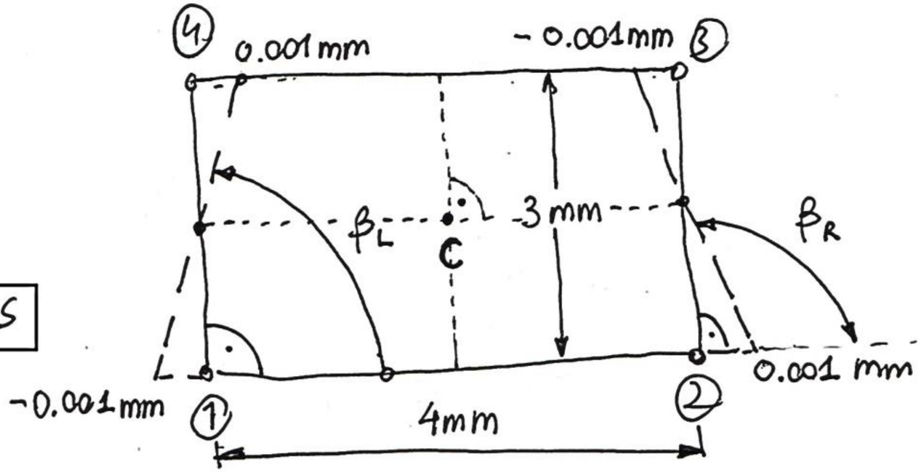


c.d. Przypadku 1. „Zginanie” w elemencie 4w

$E = 2 \cdot 10^5$ MPa

$\nu = 0.3$

$\boxed{\text{PLANE STRESS}}$



Składowe odkształcenia w elemencie:

$$\varepsilon_x^{③} = \varepsilon_x^{④} = \frac{\Delta l_{34}}{l_{34}} = \frac{-0.002\,\text{mm}}{4\ \text{mm}} = -0.5 \cdot 10^{-3}$$

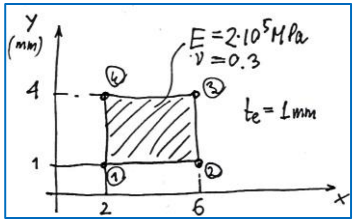
$$\varepsilon_x^{①} = \varepsilon_x^{②} = \frac{\Delta l_{12}}{l_{12}} = \frac{0.002\ \text{mm}}{4\ \text{mm}} = 0.5 \cdot 10^{-3}$$

$$\varepsilon_y^{①} = \varepsilon_y^{②} = \varepsilon_y^{③} = \varepsilon_y^{④} = 0$$

Odkształcenia postaciowe ???

$$\gamma_{xy}^{①} = \gamma_{xy}^{④} = \frac{\pi}{2} - \beta_L \cong \frac{(0.001 - (-0.001))\,\text{mm}}{3\,\text{mm}} = 0.667 \cdot 10^{-3}$$

$$\gamma_{xy}^{③} = \gamma_{xy}^{②} = \frac{\pi}{2} - \beta_R \cong \frac{(-0.001 - 0.001)\,\text{mm}}{3\,\text{mm}} = -0.667 \cdot 10^{-3}$$



Składowe naprężenia w elemencie:

$$\sigma_x^{①} = \sigma_x^{②} = \frac{E}{1-\nu^2}\left(\varepsilon_x^{①} + \nu\,\varepsilon_y^{①}\right) = \frac{2\cdot10^5\,MPa}{1-0.3^2}\cdot 0.5\cdot10^{-3} = 109.89\ MPa$$

Wyszło trochę więcej

$$\sigma_x^{③} = \sigma_x^{④} = \frac{E}{1-\nu^2}\left(\varepsilon_x^{③} + \nu\,\varepsilon_y^{③}\right) = -109.89\ MPa$$

Naprężenia na kier. Y ???

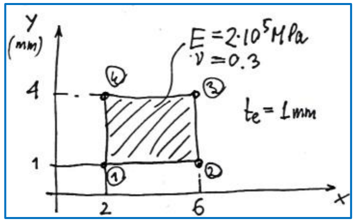
$$\sigma_y^{①} = \sigma_y^{②} = \frac{E}{1-\nu^2}\left(\varepsilon_y^{①} + \nu\,\varepsilon_x^{①}\right) = \frac{2\cdot10^5\,MPa}{1-0.3^2}\cdot 0.3\cdot 0.5\cdot10^{-3} = 32.97\ MPa$$

$$\sigma_y^{③} = \sigma_y^{④} = \frac{E}{1-\nu^2}\left(\varepsilon_y^{③} + \nu\,\varepsilon_y^{③}\right) = -32.97\ MPa$$

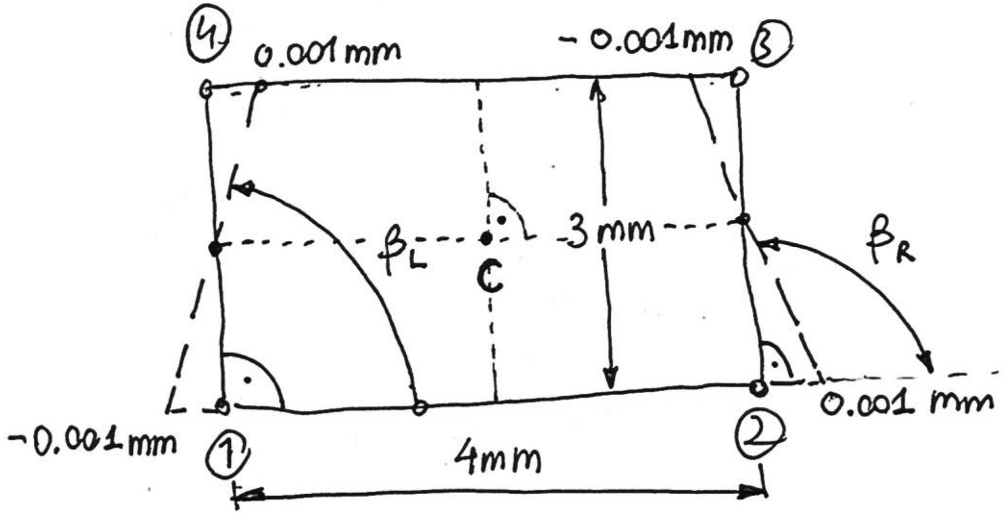
Naprężenia tnące ???

$$\tau_{xy}^{①} = \tau_{xy}^{④} = \gamma_{xy}^{①}\cdot G = 0.667\cdot10^{-3}\cdot\frac{2\cdot10^5\,MPa}{2(1+0.3)} = 51.28\ MPa$$

$$\tau_{xy}^{②} = \tau_{xy}^{③} = \gamma_{xy}^{②}\cdot G = -51.28\ MPa$$



Odkształcenia i naprężenia *__w punkcie środkowym__* *(punkt C):*

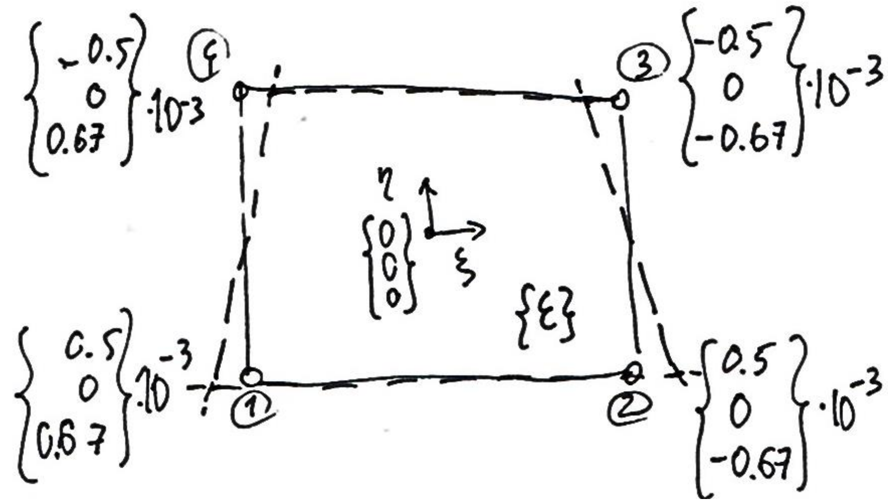


$$\varepsilon_x^C = 0, \quad \varepsilon_y^C = 0, \quad \gamma_{xy}^C = 0 \quad \Rightarrow \quad \begin{aligned} \sigma_x^C &= 0 \\ \sigma_y^C &= 0 \\ \tau_{xy}^C &= 0 \end{aligned}$$

W punkcie środkowym nie ma odkształceń ani naprężenia!

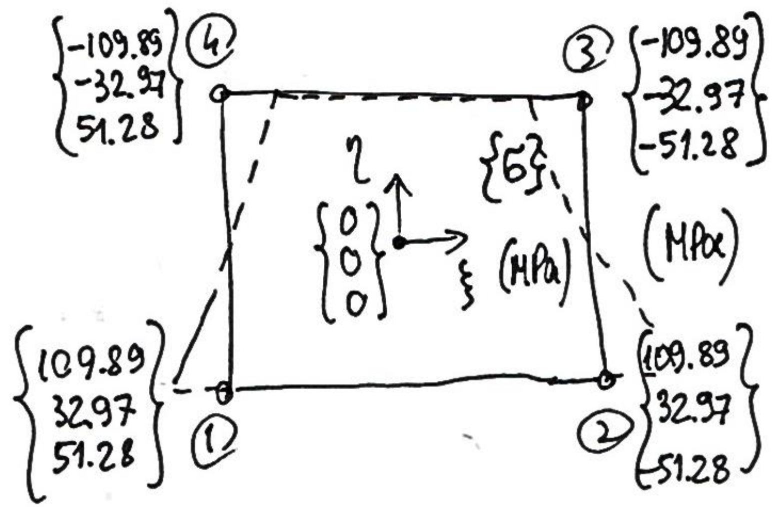
Jeśli wyliczymy odkształcenia w elemencie z macierzy odkształcenie-przemieszczenie:

$$\underset{3\times 1}{\{\varepsilon\}} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \underset{3\times 8}{\left[B(\xi,\eta)\right]} \cdot \underset{8\times 1}{\{q\}_e}$$



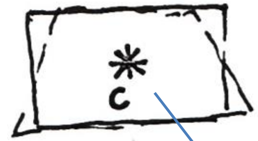
Składowe stanu naprężenia wyliczymy korzystając z macierzy stałych sprężystych:

$$\underset{3\times 1}{\{\sigma\}} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \underset{3\times 3}{[D]} \cdot \underset{3\times 1}{\{\varepsilon\}} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \cdot \underset{3\times 1}{\{\varepsilon\}}$$



Zastosujmy całkowanie numeryczne z jednym punktem Gaussa

Całkowanie numeryczne   n=1



$$W_1 \cdot W_1 = 4$$

Policzmy energię odkształcenia sprężystego w elemencie:

$$U_e = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \cdot \underset{8\times 8}{[k]_e} \cdot \underset{8\times 1}{\{q\}_e} = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \cdot \int_{\Omega_e} \underset{8\times 3}{[B]^T} \underset{3\times 3}{[D]} \underset{3\times 8}{[B]} \, d\Omega_e \cdot \underset{8\times 1}{\{q\}_e} =$$

$$= \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \cdot t_e \int_{-1}^{1}\int_{-1}^{1} \underset{8\times 3}{[B(\xi,\eta)]^T} \underset{3\times 3}{[D]} \underset{3\times 8}{[B(\xi,\eta)]} \det[J(\xi,\eta)] \, d\xi \, d\eta \cdot \underset{8\times 1}{\{q\}_e} = \left| \begin{array}{l} n=1 \\ \xi_1 = 0 \\ \eta_1 = 0 \end{array} \right| =$$

$$= \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} t_e \cdot \underset{8\times 3}{[B(0,0)]^T} \underset{3\times 3}{[D]} \cdot \underset{3\times 8}{[B(0,0)]} \cdot \det[J(0,0)] \cdot W_1 \cdot W_1 \cdot \underset{8\times 1}{\{q\}_e} \boxed{=0}$$

Energia wyszła 0 !

$$\Uparrow$$

$$\frac{\partial x}{\partial \xi} \cdot \frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi} \cdot \frac{\partial x}{\partial \eta} = 2\,\mathrm{mm} \cdot 1.5\,\mathrm{mm} - 0 \cdot 0 = 3\,\mathrm{mm}^2$$

$$\underset{3\times 3}{[D]} = \frac{2 \cdot 10^5}{1-0.3^2} \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-0.3) \end{bmatrix} \mathrm{MPa}$$

$$\underset{3\times 8}{[B(0,0)]} = \begin{bmatrix} -1/8 & 0 & 1/8 & 0 & 1/8 & 0 & -1/8 & 0 \\ 0 & -1/6 & 0 & -1/6 & 0 & 1/6 & 0 & 1/6 \\ -1/6 & -1/8 & -1/6 & 1/8 & 1/6 & 1/8 & 1/6 & -1/8 \end{bmatrix} \frac{1}{\mathrm{mm}}$$

Spróbujmy wyliczyć jeszcze raz energię sprężystą elementu inaczej:

$$U_e = \frac{1}{2}\int_{\Omega_e} \lfloor \underset{1\times 3}{\varepsilon} \rfloor \cdot \{\underset{3\times 1}{\sigma}\} d\Omega_e = \frac{1}{2} t_e \int_{-1}^{1}\int_{-1}^{1} \lfloor \underset{1\times 3}{\varepsilon(\xi,\eta)} \rfloor \cdot \{\underset{3\times 1}{\sigma(\xi,\eta)}\} \cdot \det[J(\xi,\eta)] \cdot d\xi d\eta =$$

$$= \left| \begin{array}{l} n=1 \\ \xi_1 = 0 \\ \eta_1 = 0 \\ W_1 W_1 = 4 \end{array} \right| = \frac{1}{2} t_e \lfloor \underset{1\times 3}{\varepsilon(0,0)} \rfloor \cdot \{\underset{3\times 1}{\sigma(0,0)}\} \cdot \det[J(0,0)] \cdot W_1 W_1 = \boxed{0}$$

$$\lfloor \underset{0}{\underset{\parallel}{\varepsilon_x^c}}, \underset{0}{\underset{\parallel}{\varepsilon_y^c}}, \underset{0}{\underset{\parallel}{\gamma_{xy}^c}} \rfloor \qquad \begin{Bmatrix} \sigma_x^c \\ \sigma_y^c \\ \tau_{xy}^c \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

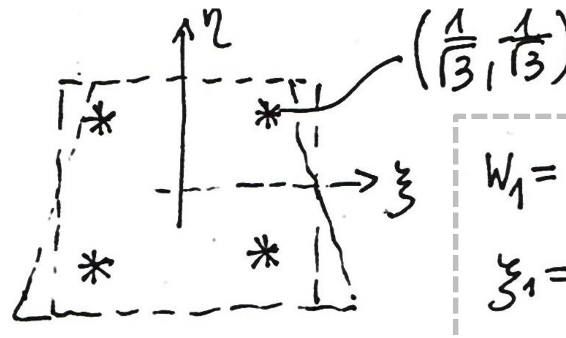
Wyszło znów 0 !

$$\Rightarrow \boxed{U_e^{\tau} = 0, \; U_e^{\sigma} = 0}$$

Jak widać energia sprężysta od naprężenia normalnego i tnącego jest zerowa

Zastosujmy całkowanie numeryczne z dwoma punktami Gaussa

Całkowanie numeryczne   n=2



$W_1 = W_2 = 1$

$\xi_1 = -\frac{1}{\sqrt{3}}$, $\xi_2 = \frac{1}{\sqrt{3}}$

$\eta_1 = -\frac{1}{\sqrt{3}}$, $\eta_2 = \frac{1}{\sqrt{3}}$

$$U_e = \frac{1}{2}\underset{1\times 8}{L q \rfloor_e} \cdot t_e \int_{-1}^{1}\int_{-1}^{1} \underset{8\times 3}{[B(\xi,\eta)]^T}\,\underset{3\times 3}{[D]}\cdot\underset{3\times 8}{[B(\xi,\eta)]}\det[J(\xi,\eta)]\,d\xi\,d\eta \cdot \underset{8\times 1}{\{q\}_e} =$$

$$\underset{8\times 8}{[f(\xi,\eta)]} = \underset{8\times 3}{[B(\xi,\eta)]^T}\cdot\underset{3\times 3}{[D]}\cdot\underset{3\times 8}{[B(\xi,\eta)]}\cdot\det[J(\xi,\eta)]$$

$$= \frac{1}{2}\underset{1\times 8}{L q \rfloor_e} \cdot t_e \int_{-1}^{1}\int_{-1}^{1} \underset{8\times 8}{[f(\xi,\eta)]}\,d\xi\,d\eta \cdot \underset{8\times 1}{\{q\}_e} =$$

Wyszło inaczej !

$$= \frac{1}{2}\underset{1\times 8}{L q \rfloor_e} t_e \cdot \left(\underset{8\times 8}{[f(\xi_1,\eta_1)]}\cdot W_1 W_1 + \underset{8\times 8}{[f(\xi_2,\eta_1)]}\cdot W_2 W_1 + \underset{8\times 8}{[f(\xi_2,\eta_2)]}\cdot W_2 W_2 + \underset{8\times 8}{[f(\xi_1,\eta_2)]}\cdot W_1 W_2\right)\cdot \underset{8\times 1}{\{q\}_e} = \boxed{0.1783\ \text{Nmm}}$$

Spróbujmy wyliczyć jeszcze raz energię sprężystą elementu inaczej (osobno dla naprężenia normalnego i osobno dla naprężenia tnącego):

$$U_e^{\sigma} = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \cdot t_e \cdot \int_{-1}^{1}\int_{-1}^{1} \underset{8\times 3}{[B_{\varepsilon}(\xi,\eta)]^T} \cdot \underset{3\times 3}{[D]} \cdot \underset{3\times 8}{[B_{\varepsilon}(\xi,\eta)]} \det[J(\xi,\eta)]\, d\xi\, d\eta \, \{q\}_e =$$

$$= 0.1099 \text{ Nmm}$$

Energia sprężysta od naprężenia normalnego

$$U_e^{\tau} = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \, t_e \cdot \int_{-1}^{1}\int_{-1}^{1} \underset{8\times 3}{[B_{\gamma}(\xi,\eta)]^T} \underset{3\times 3}{[D]} \cdot \underset{3\times 8}{[B_{\gamma}(\xi,\eta)]} \det[J(\xi,\eta)]\, d\xi\, d\eta \cdot \{q\}_e =$$

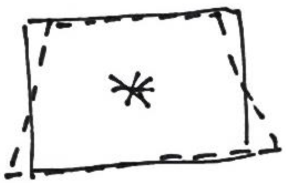
$$= 0.0684 \text{ Nmm}$$

Energia sprężysta od naprężenia tnącego

Energia odkształcenia sprężystego (porównanie dla różnej liczby punktów całkowania”

Całkowanie numeryczne

$n = 1$



$w_1 w_1 = 4$

$$U_e = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \underset{8\times 8}{[k]_e} \cdot \underset{8\times 1}{\{q\}_e} = 0$$

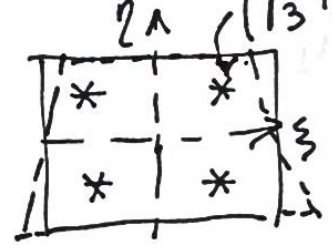
$$U_e^{\sigma} = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \underset{8\times 8}{[k_{\varepsilon}]_e} \underset{8\times 1}{\{q\}_e} = 0$$

$$U_e^{\tau} = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \underset{8\times 8}{[k_{\gamma}]} \{q\}_e = 0$$

Postać o zerowej energii
(tzw. „hourglassing” - „klepsydrowa”)

$n = 2$

$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$



$W_1 = W_2 = 1$

$U_e = 0{,}1783 \quad Nmm$

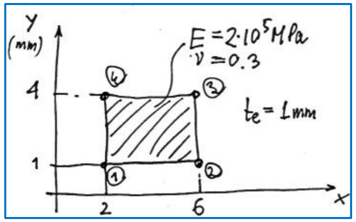
$U_e^{\sigma} = 0{,}1099 \quad Nmm$

$U_e^{\tau} = \boxed{0{,}0684 \quad Nmm}$

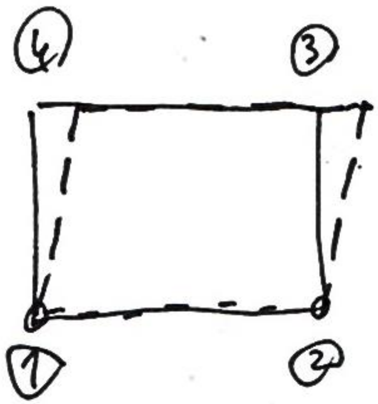
Niezerowa energia naprężenia tnącego!

$(U_e^{\tau} = 38\% \cdot U_e \ ?)$

Element jest bardziej sztywny
(tzw. „shear locking”)



***Przypadek 2**. Rozważmy deformację „Ścinanie”*



$$\underset{\substack{8\times1 \\ (mm)}}{\{q\}_e} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.001 \\ 0 \\ 0.001 \\ 0 \end{Bmatrix} \begin{matrix} \\ \\ \\ \\ (u_3) \\ \\ (u_4) \\ \\ \end{matrix}$$

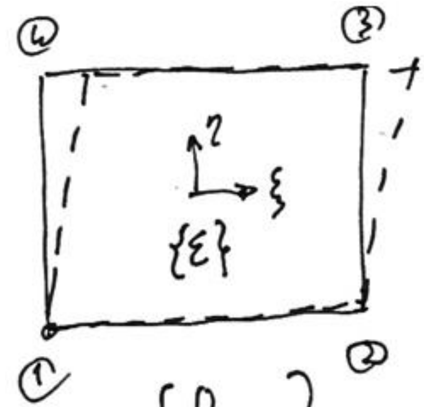
Oszacowanie stanu odkształcenia i naprężenia:

$$\gamma_{xy} \cong \frac{0.001\,mm}{l_{23}} = \frac{0.001\,mm}{3\ mm} = 0.33\cdot10^{-3}$$

$$\tau_{xy} = G\cdot\gamma_{xy} = \frac{E}{2(1+\nu)}\cdot\gamma_{xy} = 25.64\,MPa$$

Wyliczmy odkształcenia w elemencie z macierzy odkształcenie-przemieszczenie:

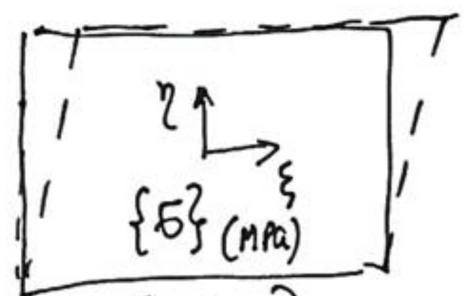
$$\underset{3\times 1}{\{\varepsilon\}} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \underset{3\times 8}{[B(\xi,\eta)]} \cdot \underset{8\times 1}{\{q\}_e}$$



$$\begin{Bmatrix} 0 \\ 0 \\ 0.33\cdot 10^{-3} \end{Bmatrix}$$ - *stałe*

Składowe stanu naprężenia wyliczymy korzystając z macierzy stałych sprężystych:

$$\underset{3\times 1}{\{\sigma\}} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \underset{3\times 3}{[D]} \cdot \underset{3\times 1}{\{\varepsilon\}} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}$$

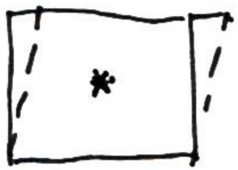


$$\begin{Bmatrix} 0 \\ 0 \\ 25.641 \end{Bmatrix}$$ - *stałe*

Energia odkształcenia sprężystego (porównanie dla różnej liczby punktów całkowania)

Całkowanie numeryczne

$n=1$



$w_1 w_4 = 4$

$$U_e = \frac{1}{2} \lfloor q \rfloor_e \cdot [k]_e \cdot \{q\}_e = 0.0513 \text{ Nmm}$$

$$U_e^{\sigma} = 0 \quad , \quad U_e^{\tau} = U_e$$

$n=2$

$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

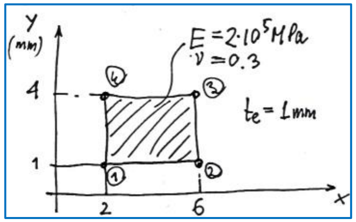
$w_1 = w_2 = 1$

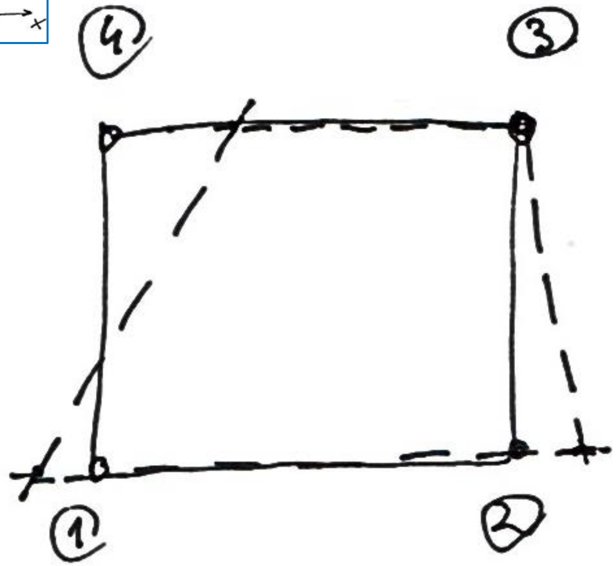
$$U_e = 0.0513 \text{ N mm}$$

$$U_e^{\sigma} = 0 \quad , \quad U_e^{\tau} = U_e$$

Wartość identyczna niezależnie
od liczby punktów Gaussa

Przypadek 3. *Rozważmy deformację „Zginanie + Ścinanie” (superpozycja 1+2)*



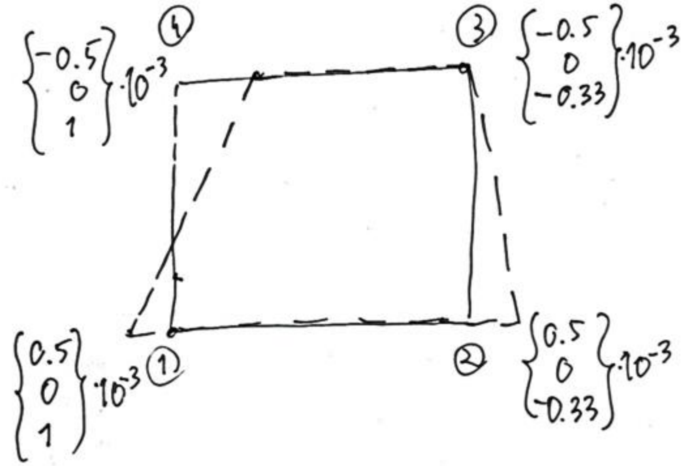


$$\underset{8\times 1}{\{q\}_e} = \begin{array}{c} \begin{Bmatrix} -0.001 \\ 0 \\ 0.001 \\ 0 \\ 0 \\ 0 \\ 0.002 \\ 0 \end{Bmatrix} \end{array} \begin{array}{l} (u_1) \\ \\ (u_2) \\ \\ \\ \\ (u_4) \\ \\ \end{array}$$

(mm)

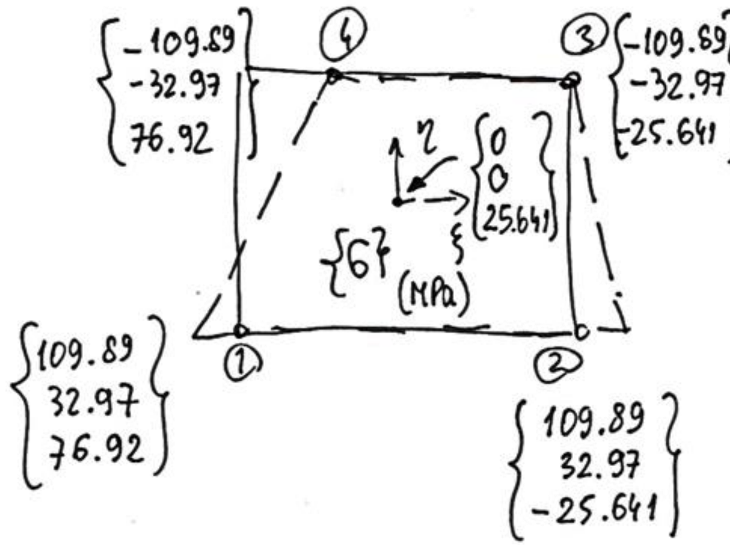
Wyliczmy odkształcenia w elemencie z macierzy odkształcenie-przemieszczenie:

$$\underset{3\times1}{\{\varepsilon\}} = \begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix} = \underset{3\times8}{[B(\xi,\eta)]}\cdot \underset{8\times1}{\{q\}_e}$$



Składowe stanu naprężenia wyliczymy korzystając z macierzy stałych sprężystych:

$$\underset{3\times1}{\{\sigma\}} = \begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \underset{3\times3}{[D]}\cdot\underset{3\times1}{\{\varepsilon\}} = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1}{2}(1-\nu)\end{bmatrix}$$



Energia odkształcenia sprężystego (porównanie dla różnej liczby punktów całkowania)

Całkowanie numeryczne

$n = 1$

$$U_e = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \underset{8\times 8}{[k]_e} \underset{8\times 1}{\{q\}_e} = 0.0513 \text{ Nmm}$$

$$U_e^{\sigma} = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \underset{8\times 8}{[k_\varepsilon]} \{q\}_e = 0 \text{ Nmm}$$

$$U_e^{\tau} = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} [k_\gamma] \{q\}_e = 0.0513 \text{ Nmm} = U_e$$

$n = 2$

$$U_e = 0.22955 \text{ Nmm}$$

$$U_e^{\sigma} = 0.1099 \text{ Nmm}$$

$$U_e^{\tau} = 0.1197 \text{ Nmm}$$

$$U_e^{\tau} = 52\% \, U_e$$

$$U_e^{\tau}(case3) = U_e^{\tau}(case1) + U_e^{\tau}(case2) = (0.0684 + 0.0513) \text{ Nmm} = 0.1197 \text{ Nmm}$$

„shear locking”

Podsumowanie

CASE U_e [Nmm]	n=1			n=2		
	U_e^σ	U_e^τ	U_e	U_e^σ	U_e^τ	U_e
1. „BENDING"	0	0	0	0.1099	0.0684	0.1783
2. „SHEAR"	0	0.0513	0.0513	0	0.0513	0.0513
3. „BENDING + SHEAR"	0 = (0+0)	0.0513 = (0+0.0513)	0.0513 (0+0.0513)	0.1099 = (0+ 0.1099)	0.1197 = (0.0684+ 0.0513)	0.22955 = 0.0513+ 0.1783

„hourglassing"

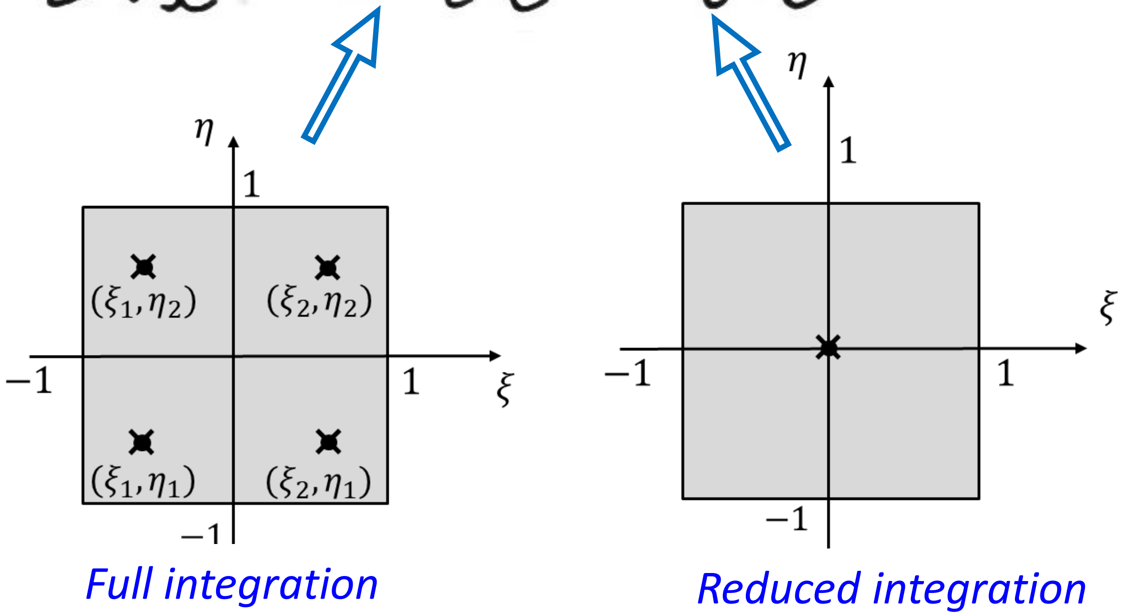
„shear locking"

Co zrobić aby poprawić wyniki?

$$[k]_e = [k_\varepsilon]_e + [k_\gamma]_e$$

Wnioski:

(technologia elementu)



$$[k_\varepsilon]_e = t_e \int_{-1}^{1}\int_{-1}^{1}\left([B_\varepsilon]^T[D][B_\varepsilon]\cdot \det[J(\xi,\eta)]\right) d\xi\, d\eta$$

$$[k_\gamma]_e = t_e \int_{-1}^{1}\int_{-1}^{1}\left([B_\gamma]^T[D][B_\gamma]\det[J(\xi,\eta)]\right) d\xi\, d\eta$$

Mieszana reguła kwadratur

Full integration ($n = 2$) :

$$U_e^{\sigma} = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} \underset{8\times 8}{[k_{\varepsilon}]} \{q\}_e = 0.1099 \text{ Nmm}$$

Reduced integration ($n = 1$) :

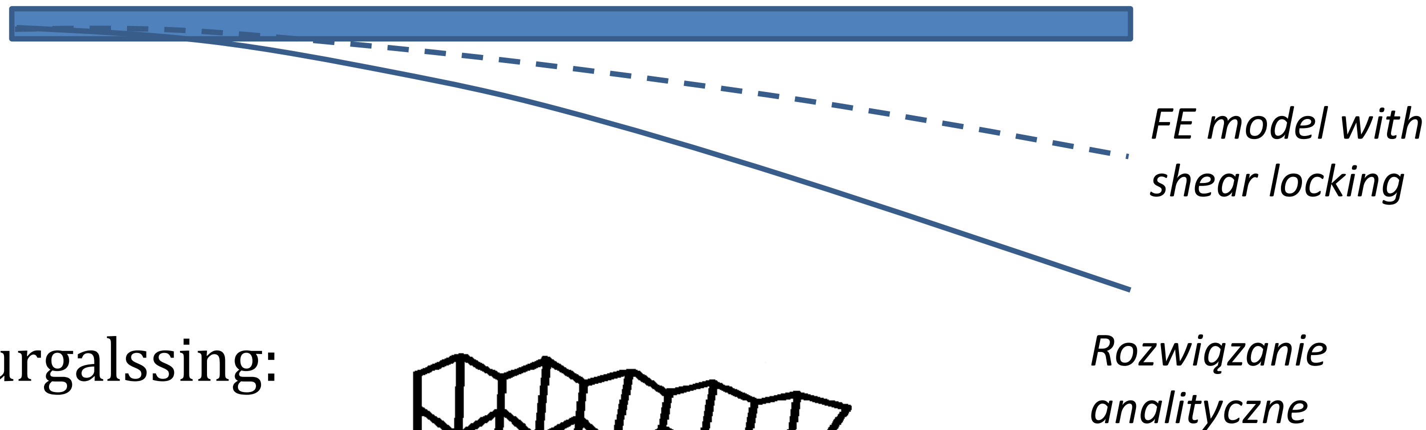
$$U_e^{\tau} = \frac{1}{2} \underset{1\times 8}{\lfloor q \rfloor_e} [k_{\gamma}] \{q\}_e = 0.0513 \text{ Nmm}$$

$$= U_e$$

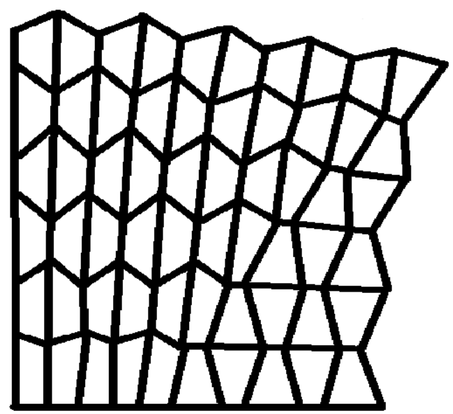
$$U_e = U_e^{\sigma} + U_e^{\tau} = 0.16117 \text{ Nmm}$$

$$U_e\,(\text{case}3) = U_e^{\sigma}(\text{case}1) + U_e^{\tau}(\text{case}2)$$

– shear locking:



– hourgalssing:



– volumetric locking in nearly incompressible materials ($\nu \cong 0.5$)

Technologia elementu – materiały liniowe

Element	**Stress State**	**Poisson's ratio <= 0.49**	**Poisson's ratio > 0.49 (or anisotropic materials)**
PLANE182	Plane stress	KEYOPT(1) = 2 (Enhanced strain formulation)	KEYOPT(1) = 2 (Enhanced strain formulation)
	Not plane stress	KEYOPT(1) = 3 (Simplified enhanced strain formulation)	KEYOPT(1) = 2 (Enhanced strain formulation)
PLANE183	Plane stress	No change	No change
	Not plane stress	No change	No change
SOLID185		KEYOPT(2) = 3 (Simplified enhanced strain formulation)	KEYOPT(2) = 2 (Enhanced strain formulation)
SOLID186		KEYOPT(2) = 0 (Uniform reduced integration)	KEYOPT(2) = 0 (Uniform reduced integration)
SHELL281		No change	No change

(+dodatkowe funkcje kształtu)

Shear Locking and Hourglassing in MSC Nastran, ABAQUS, and ANSYS

Eric Qiuli Sun

Abstract

A solid beam and a composite beam were used to compare how MSC Nastran, ABAQUS, and ANSYS handled the numerical difficulties of shear locking and hourglassing. Their tip displacements and first modes were computed, normalized, and listed in multiple tables under various situations. It was found that fully integrated first order solid elements in these three finite element codes exhibited similar shear locking. It is thus recommended that one should avoid using this type of element in bending applications and modal analysis. There was, however, no such shear locking with fully integrated second order solid elements. Reduced integration first order solid elements in ABAQUS and ANSYS suffered from hourglassing when a mesh was coarse. If there was only one layer of elements, the reported first mode of the beam examples from ABAQUS and ANSYS was excessively smaller than the converged solutions due to hourglassing. At least four layers of elements should, therefore, be used in ABAQUS and ANSYS. MSC Nastran outperformed ABAQUS and ANSYS by virtually eliminating the annoying hourglassing of reduced integration first order 3D solid elements because it employed bubble functions to control the propagation of non-physical zero-energy modes. Even if there was only one layer of such elements, MSC Nastran could still manage to produce reasonably accurate results. This is very convenient because it is much less prone to errors when using reduced integration first order 3D solid elements in MSC Nastran.

https://moodle.umontpellier.fr/pluginfile.php/480056/mod_resource/content/0/Sun-ShearLocking-Hourglassing.pdf